When the collective acts on its components: economic crisis autocatalytic percolation

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Abstract. Agent-based models have improved the standards for empirical support and validation criteria in social, biological, cognitive and human sciences. Yet, the inclusion, in these models, of vertical interactions between various aggregation levels remains a challenge. We study analytically, numerically and by simulation the generic consequences of interactions between the collective and its individual components:

- the appearance of an autocatalytic loop between the dynamics of the collective and its components;
- the system, which is dominated by a limited number of factors amplified by this collective↔individuals autocatalytic loop;
- the microscopic features, which are not involved in the autocatalytic loop and are irrelevant at the systemic level; and
- how the above clarify the interplay between macroscopic predictable features and the ones dependent on random unpredictable individual events.

Using the social and market percolation framework, we study the dramatic effects of the collective↔individuals autocatalytic loop on economic crisis propagation:

- the percolation transition becomes discontinuous;
- there are a few relevant regions and regimes corresponding to a quite diverse range of response policy options;
- there are stability ranges where appropriate policies can help to avoid macroscopic crisis percolation; and
- beyond those regions the systemic crisis might become unstoppable.

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1. Introduction

In this paper, we study collective processes that have the property that they enhance the very individual behavior that generated them. This generates an autocatalytic loop between the collective process and its individual components. Thus, such processes have a decisive advantage and dominate at the macroscopic scale. We argue that such processes are responsible for many of the sudden changes that threaten our climate, the ecology of our environment, our social order and the stability of our economies around the world. In a series of works (Goldenberg *et al* 2000, Solomon *et al* 2000, Weisbuch *et al* 2001, Aleksiejuk and Holyst 2001, Bornholdt 2001, Erez *et al* 2005, Goldenberg *et al* 2005, Yaari *et al* 2006, Erez *et al* 2007, Sieczka and Holyst 2009, Lorenz *et al* 2009), the elementary concepts to express, understand and steer such emergent phenomena have been introduced. Phenomena such as 'word-of-mouth', 'domino failures' and 'viral products' have been shown to be crucial in explaining, forecasting and controlling massive global (systemic) changes, such as electric power blackouts, WWW virus attacks, and economic, financial and credit crises.

In particular, Antonelli (1996), Goldenberg *et al* (2000), Erez *et al* (2005), Goldenberg *et al* (2005), Yaari *et al* (2006) and Erez *et al* (2007) have studied the propagation of knowledge, products, opinions and information throughout social and economic networks using the social and market percolation framework. However, in order to describe more faithfully real economic systems, the original percolation framework has to be enlarged to account for economic facts that have no analogue in physical systems. In particular, as opposed to physical systems where the dynamics are usually individual↔individual, in social and economic systems there are significant collective→individual and individual→collective effects, such as government interventions, the general state of the economy, the 'mood of the

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market' and mass media. Such effects have been shown in the past to lead to self-organized percolation criticality (Solomon *et al* 2000), different stages of technological development along the product lifecycle (Frenken *et al* 2008), long tails at the early stages of innovation diffusion (Hohnisch *et al* 2008) and dependence of demand-pull policies on the learning curve (Cantono and Silverberg 2009).

Here, we take a further step towards integrating those effects. We show that they can generate quite generically powerful autocatalytic loops capable of promoting individual behavior to collective discontinuous changes that affect, suddenly and dramatically, the entire system. We single out for study the class of systems possessing a 'collective–individual autocatalytic loop'. By this we mean that the collective as such acts on its own individual components in a way that enhances the very individual behavior that generated it. Consequently, the collective and the individual behavior will enhance one another. Processes with this property will have a decisive advantage over processes that do not have it. As a consequence, most of the collective phenomena that are observed at the macroscopic scale have this property. By selecting at the macroscopic level the features that have the collective–individual autocatalytic property, this mechanism reduces greatly the number of microscopic features necessary to describe the system at macroscopic scales and increases its predictability.

In this paper, we try to strike a balance between the generality of the collective–individual autocatalytic loop and the necessity to give a specific embodiment of the ideas. To this effect, we have chosen to use terms specific to the economic context and, in particular, to the current financial crisis.

This systemic worldwide event has challenged in an acute way some of the basic postulates that mainstream macroeconomics is built upon (*The Economist* 2009a, 2009b, 2009c). The cascade of events that disturbed the recent course of the world economy has very little to do with the scenario prescribed by any of the market theories (Lux and Westerhoff 2009). In fact, as stated in Colander *et al* (2008), even after the reality of the crisis was recognized, there was little that the economics profession had to say about the expected outlook and/or about the recommended measures: most of the questions fell outside the range addressable in the traditional economic conceptual framework (Leijonhufvud 2009). In order to understand and control the evolution of mass phenomena such as the current crisis, one has to give up the concept of linear causality chains, which associates to each effect a cause. One has to think, rather, in terms of a collective, emergent causality where a complex network of interactions between many autonomous agents and their collective as such can lead to collective phenomena that are completely different from the intentions, scales and scope of the individual components.

In this context, the individual→collective and collective→individual effects are represented, respectively, by the following:

- the global influence that the distress of one economic agent (firm/bank) has on the general state of the economic system (e.g. resilience of the system to liquidity shortages, the system's exposure to risk, general availability of credit, etc.) beyond the direct influence that it has on its business associates;

- in turn, the global status of the economy has a direct effect on individual agent sensitivity to its partner associates' distress.

The main result of our model is that there exists a discontinuous phase transition between localized and globalized/percolated distress regimes. This phase transition is governed by four critical factors.

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More precisely, we will find that, for a given status of the system, in order for the distress to propagate throughout the entire system, one needs the following:

- an initial density of fragile economic agents (firms, banks, financial companies) that is larger than a certain critical value; and/or
- a critical strength of the initial exogenous shock (measured by the number of failures induced exogenously to otherwise healthy economic agents); and/or
- a critical duration of the exogenous pressure (measured by the number of contagions induced exogenously to otherwise healthy economic agents); and/or
- a critical strength of the individual→collective effect (measured by the exponent of the dependence of the global economic index on the number of distressed agents).

As long as the density of fragile agents is low, only extremely serious exogenous shocks can lead to macroscopic (systemic) effects. Once the critical initial density is reached, even small shocks can lead to the collapse of large parts of the system unless they are stopped by intervention in the relevant early propagation stage.

In order to address these issues quantitatively, our model formalizes mathematically one of the most fascinating features of social and economic systems: the interplay between individual→collective, collective→individual and individual↔individual effects. This allows us to evaluate the critical number of fragile agents capable of leading to a macroscopic disaster, how this will propagate and when and how (if at all) it can be stopped. The presence of an autocatalytic loop between the individual and collective levels profoundly affects the character of the usual percolation transition, rendering the system dynamics discontinuous, path dependent and irreversible. However, it also introduces additional features, such as periods of contagion slow-down, stability ranges where the systemic collapse can be still avoided, and hang-up points where small and almost costless intervention can save the entire system.

The present autocatalytic percolation framework extends beyond the crisis study to an entire range of other social and economic systems. It allows one to predict, control and offer decision making tools for harnessing a variety of desirable and undesirable contagion processes.

The structure of the paper is as follows. In section 2.1, we give the economic background and motivation. In section 2.2, we give a detailed formulation of the connection between the crisis phenomenology and the agent-based *social and market percolation framework*. In section 2.3, we describe the crisis propagation scenario and preview the main results. In section 2.4, we write explicitly the iterative equations characterizing the *autocatalytic percolation process*. In sections 3.1 and 3.2, we study, by theoretical and analytical means, the process introduced in section 2. In section 4.1, we describe in detail the numerical simulations of the agent-based model. In section 4.2, we address the effects of randomness on individual realizations of the process and introduce the concept of *lower envelope*. In section 4.3, we confront in detail the theoretical and simulation results. In section 5, we discuss the stability ranges of the system and its policy implications. In section 6, we offer conclusions on the actual economic systems and the outlook.

2. General conceptual framework

2.1. Economic distress contagion in terms of autocatalytic percolation

The mechanism demonstrated in the present paper describes and explains the emergence of complex, collective macroscopic phenomena in a wide range of domains: ecology, social

science, cognition, etc. In the simplest formulation it exploits the influence of the collective as such on its individual components. While systems of this type have been described in the past, the present framework allows a precise formulation of the collective–individual autocatalytic loop mechanism and a rigorous analytical, numerical and simulation treatment that leads to predictions that are verifiable by empirical experiments and observations. We thus propose a sketch of a possible future line of applied research that has to be implemented in detail in order to confront quantitatively the economic reality in general and the current financial crisis in particular. Such an effort to translate its beautiful theoretical generality would surely be most appreciated by practitioners, policy makers and applied scientists.

We claim that several main causes contributed to the current economic crisis: underestimation of systemic risk, systemic insolvency and liquidity shortage were the reasons for the sudden systemic collapse. Scholars who are perceived to be mainstream economists gave different explanations about financial crisis in general (Lucas 1972, Friedman 1981, Bernanke 1983), and the causes of the current one in particular (Krugman 2009, Blanchard 2009, O'Grady 2009—interview to Gary Becker). Moreover, there is no consensus on what seems to us to be the most realistic alternative: that is the Hyman Minsky theory of financial fragility (Davidson 2008).

We claim that one of the problems with these macro-economic explanations is that they predict immediate and automatic diffusion of financial distress throughout the entire economic system. In reality, not every financial 'event' spreads over the entire system (Kaminsky *et al* 2003). A cascade of failures depends not only on the power of the amplification mechanism but also on the strength of the shock affecting the initially distressed agents (Dow 2000). The explanation of the delayed, non-automatic and sometimes disproportionate effects relies on taking into account the barriers to the overall diffusion of distress due to the macro- and micro-structure of the system as well as the autocatalytic loops between the scales. A small distress may trigger the system to a crisis, but the system's resilience depends on both individual→collective and collective→individual effects, as well as individual↔individual interactions.

In the case of the current crisis, the seeds for the critical collective unstable state were already sown at the beginning of 2000 during the dot-com crisis and the consequent crash of the individual stock market indexes (DeLong and Magin 2006). This was clearly an example of a collective→individual induced effect since the individual firm failures often had little to do with their actual worth. The actions taken by the Federal Reserve (Fed), along with high savings rates mainly from Asia but also from the rest of the world (O'Grady 2009—interview to Gary Becker), triggered a drop in interest rates. This was part of a series of measures that affected the system at the collective macroscopic level. From 2001 to 2003, the Fed slashed the federal fund target in an attempt to 'jumpstart the economy out of recession' (Murphy 2008). The credit market inflated to a monstrous size, unfettered by the extremely lax requirements for accessing credit. This reflected upon the individual firm level that accumulated vastly more debt compared to the assets to guarantee it. As the housing bubble exploded, the propagation of distress became 'autocatalytic': not only did the individuals and banks experiencing losses transmit their distress to their financial/business partners, but also the deterioration of their own collaterals decreased the worth of the assets leveraging the financial liabilities of everybody else as a collective. This bottom-up transformation of individual distress into collective systemic negative mood in turn negatively affected each of the individual economic players.

Moreover, via such autocatalytic ('procyclical') collective↔individual loops, the misalignment between the cost of financial intermediations and the associated risk (through

non-regulated chains of collateralized debt obligations) led ultimately to a loss of control on the intermediations themselves. This collective state impacted well beyond the expected degree upon the individual probability of insolvency and on the system fragility on all scales: even though individual financial institutions considered themselves hedged against any particular risk, they were in fact markedly exposed to systemic/collective risk. Individual exposure does not always reveal the collective exposure of the whole system in as far as it ignores feedback loops connecting the individual players to their collective.

Indeed, a 'vertical' (collective→individual) effect by which the totality of individual players affects each individual player indirectly by the general depreciation of all the assets could not even be expressed by the usual macro-economic models in the same way in which the macroeconomic models could not express the 'horizontal' ('peer-to-peer') individual→individual contagion of distress between business partners. The crisis autocatalytic loop was closed by recurrent actions through which individuals were affecting the collective state of the system. For instance, during the beginning of the race for liquidity, one observed a rush of withdrawals, precautionary increases in the reserve/deposit ratios and an increased desire by banks for very liquid or rediscountable assets. Helped by this collective–individual autocatalytic loop, the individual→individual 'horizontal' contagion through business/financial interactions spread the distress that was originated by the sub-prime mortgage crisis to other parts of the economy and even to other countries, destabilizing them in turn. In particular, European banks and European public authorities who were holders of packaged mortgage-backed assets underwritten by American financial institutions saw the rise by contagion of their credit risk and the decrease of their share prices. Again, this peer-to-peer distress propagation was promoted to systemic scales by bottom-up mechanisms and further propelled by ensuing top-to-down effects: systemic risk and systemic insolvency. This extended the crisis far away from the initial centers of infection and from the obvious weak individuals to the rest of the, until then, 'safe' elements of the system.

2.2. Detailed formulation of the problem

Let us express in a more formal way the above ideas:

- The current status of the economy is measured by a global economic variable p.
- The status of the economy at the start of the crisis is labeled by the value $p = p_0$ of this global index when the first bankruptcy[5] takes place.
- The set of firms[6] in the economic system under study is indexed by a label $i = 1, 2, 3, \ldots, M$. We consider M 'large enough' and do not insist, in the present paper, on the effects of finite system size M.
- Each firm's fragility is characterized by a fragility threshold p_i. If the global economic index p decreases below this threshold $p < p_i$, i is said to become a *potential failure*[7] with the implications detailed below.

[5] We will call the economic agent in distress 'bankrupt'. The term 'bankrupt' as it is used throughout this paper is an abstract generic label for a range of real-life situations in which an economic agent (firm/bank) displays objective signs of distress (such as shrinking of production or of manpower, liquidity constraints, delays in payments, negative warnings to the stock market and cancelations of dividend payments).

[6] We will generically call our agents 'firms'.

[7] We define the concept of 'potential failure' or 'potentially bankrupt' to characterize the firms that are susceptible to becoming bankrupt by being infected/contaminated by an eventual bankrupt neighbor.

- Each firm i is connected to business partners or 'neighbors' j. The links (i, j) form a network with a certain geometry. This network geometry characterizes the individual↔individual interactions and is one of the main characteristics of the system.

- A firm i will go bankrupt if, and only if, two conditions are both fulfilled:
  A. one of its partners/neighbors goes bankrupt and
  B. the firm i is a potential failure, $p_i > p$.

Conditions **A** and **B** imply the possibility that, following an initial number N_0 of exogenously induced bankruptcies, the process may continue self-sustained by the iterative contagion and bankruptcy of further firms. If the number of bankruptcies reaches values of the order of M, we say that the bankruptcy wave (crisis) has percolated.

In order to perform a concrete quantitative analysis of this general phenomenon, we make the following specific assumptions.

We assume that each firm i is linked to four 'neighbors' j in a two-dimensional (2D) square lattice geometry. We consider this geometry for definiteness and in order to exploit the extensive knowledge accumulated in the percolation literature for this particular case. In real applications, *the individual↔individual network structure* will have to be input empirically. As it will turn out, the lattice choice affects the dynamics mainly through its critical percolation density and the critical (susceptibility) exponent.

We assume that the p_is are random, independent variables distributed by a Pareto power law probability distribution:

$$P(p_i > \theta) = \theta^{-\mu} \quad \text{for } \theta > 1 \quad \text{and} \quad P(p_i > \theta) = 1 \quad \text{for } \theta < 1.$$

We choose a power law distribution for the firm's fragility indices p_i because most of the measures of company performance, both positive (Axtell 2001) and negative (Fujiwara 2004, Delli Gatti *et al* 2004), have been found to display such power law distributions.

According to this assumption, the density of firms that are potential failures at an economic state characterized by the global index p is

$$\rho(p) = p^{-\mu} \quad \text{for } p > 1 \quad \text{and} \quad \rho(p) = 1 \quad \text{for } p < 1. \tag{1}$$

Note that, with this convention, there are no firms more stable than $p_i = 1$, which is reasonable if we take, by convention, $p = 1$ as the state of total stagnation of the entire economy. The exact form of equation (1) is, however, not crucial for the qualitative behavior of our system (an exponential would give similar results).

Equation (1) expresses the *collective→individual* influence of the global status p on the 'potentially bankrupt' status of the individual firms, and thus also defines implicitly the percolation critical value of the global economic index p:

$$p_c = \rho_c^{-1/\mu},$$

where ρ_c is the critical percolation density. More precisely for $\rho > \rho_c$, the potential failures form a macroscopic 'giant cluster' (in addition to many smaller ones). Thus, for $p < p_c$ there exists a macroscopic giant cluster of potential bankruptcies. If the macroscopic 'giant cluster' corresponding to a certain value $p < p_c$ happens to contain one or more of the initial N_0 bankruptcy seeds, then one is guaranteed that all the sites in this macroscopic cluster will go bankrupt.

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We assume that the bankruptcy of N_b individual firms impacts on the general status of the economy:

$$p(N_b) = p_0 N_b^{-\alpha}, \tag{2}$$

where α is the *individual→collective* interaction exponent. We take the values of α between 0.1 and 1. Below this range the influence is quite negligible and above it the influence is too brutal and unrealistic. Since $N_b = 0$ does not make sense in formula (2), we normalize the state of the system by p_0 corresponding to the initial status of the economy at the beginning of the crisis, when the first firm goes bankrupt, i.e. for $N_b = 1$.

Equation (2) has been used in the past to analyze the dependence of demand-pull policies on the learning curve (Cantono and Silverberg 2009). In our current theoretical formulation, the individual↔collective interaction exponent α expresses the capacity of the system to react and absorb the implications of the bankruptcy of its individual elements. Equation (2) implies that the first bankruptcies have a more important effect (because of the novelty and surprise), while the effect per failed firm decreases as the number of failures increases (due to habituation/desensitization). To our knowledge, a similar mechanism has been studied in the social psychology domain (see Nowak *et al* 1990). As mentioned in section 2.1, a detailed implementation will be needed in the future in order to confront it quantitatively with economic reality.

Indeed, the exact functional form in which N_b influences p is still an open social and economic problem. However, for the generic phenomena of the present paper to take place (e.g. for the continuous percolation transition to turn into a discontinuous one), it is sufficient that p in (2) is a decreasing function of N_b.

In fact the inclusion of an 'individual→collective' effect of the type expressed by equation (2) is the crucial ingredient in the present model. In the absence of equation (2), the dynamics of the process would be rather trivial: the propagation of the bankruptcy crisis to macroscopic level would only depend on whether the initial economic index p_0 is smaller ($p_0 < p_c$) or larger ($p_0 > p_c$) than the usual (static) critical percolation value p_c. Moreover, the usual percolation phase transition at $p_0 = p_c$ would be continuous. In the presence of equation (2), the new critical value p_{0c} separating the stable phase from the systemic crisis phase is significantly larger than p_c, and the phase transition at p_{0c} is discontinuous.

Why the individual→collective mechanism should take the form of a power law decay is still an open question that surely needs and deserves further scientific inquiry.

Obviously, rules A and B together with equation (2) provide a mechanism for the crisis propagation but do not provide a mechanism for initiating/triggering a crisis. We still have to give a formal description of the exogenous mechanisms that generate the initial bankruptcies that trigger the crisis, i.e. the first wave of bankruptcies has to take place by a different mechanism than rules A and B. Following the empirical evidence, we assume that the initial bankruptcies may take place in two phases:

Phase I. N_0 sites/firms that go bankrupt at this stage. These sites are chosen randomly and completely independent of their positions or their $p_i s$.

Phase II. If phase I is not enough to produce a 'percolation' of the crisis, one may consider an additional '*forced contagion*' phase that increases exogenously the number of bankruptcies to N_f. These additional bankruptcies are no longer completely random and take place according to the **A + modified B** rules:

- **A-(unchanged)**: only firms that have a bankrupt neighbor can go bankrupt;

- **B-modified/relaxed**: as long as $N_b < N_f$, the most fragile site i having a bankrupt neighbor goes bankrupt even if its $p_i < p$. This particular way of exogenously forcing the bankruptcy contagion is not the only possible one. Other alternatives for increasing the bankruptcy clusters to $N_b = N_f$ would work the same way. We made this particular choice because of its similarity to the already known invasion percolation procedure (Wilkinson and Willemsen 1983).

For a given network geometry characterized by a critical percolation density ρ_c and a critical exponent γ (see section 2.4, equation (3)), a given individual→collective interaction exponent α, a given collective→individual exponent μ and a given initial status of the economy p_0, we are now in the position to ask: what is the severity of the exogenous shock (in terms of N_0 and N_f) sufficient to ensure that the contagion process continues to propagate/percolate (according to rules **A** and **B**) to a macroscopic crisis?

2.3. The dynamic process and main results

Let us make the question above more precise by making explicit the scenario that we wish to eventually formulate and analyze quantitatively in the rest of the paper.

(a) Following the exogenously driven phases I and II, the bankruptcy clusters total N_f firms (if phase II is absent, the only implication for the following analysis is that $N_f = N_0$). According to equation (2), the economic index is then

$$p(N_f) = p_0 N_f^{-\alpha}.$$

According to equation (1), this means a density of potential failures (density of sites with $p_i > p$) equal to

$$\rho(N_f) = p(N_f)^{-\mu} = p_0^{-\mu} N_f^{\alpha\mu}.$$

(b) Thus there is a significant probability that, even after the exogenous factors cease to act, some of the firms that are neighbors of the bankrupt clusters may be potential failures and go bankrupt as soon as rules **A** and **B** start being enforced.

(c) Consequently, the bankrupt clusters grow and more potential failures become neighbors of the bankrupt clusters and go bankrupt themselves, according to **A** and **B**, and so on, iteratively.

(d) This will increase the number of bankruptcies to a higher level $N_1 > N_f$, which will bring the economic index p to an even lower level $p(N_1) = p_0 N_1^{-\alpha} < p(N_f)$.

(e) That, in turn, may bring further sites (firms) to fulfill $p_i > p(N_1)$ and become potential failures.

(f) Similar to step (b), the ones among those potential failures that are neighbors of one of the bankrupt clusters will actually go bankrupt, according to the **A** and **B** rules.

(g) Similar to (c), due to the new bankruptcies, the clusters grow and more potential failures become neighbors of the bankrupt clusters and go bankrupt themselves, according to the **A** and **B** rules, and so on.

(h) Similar to (d), this would further increase the number of bankruptcies to N_2, which, according to equation (1), would in turn lower the index to $p(N_2) = p_0 N_2^{-\alpha} < p(N_1)$, and so on.

For some values of the parameters p_0, α, N_0, N_f, μ, the iterative bankruptcies contagion process described above might eventually stop at a finite value of N. For other values of the parameters, the crisis might propagate throughout the entire system. In the detailed theoretical, numerical and Monte Carlo simulation analysis below, we will find sharp discontinuous phase transitions between these two extreme regimes in all the parameters.

However, due to the symmetries of the problem,

$$N_b(N_0, p_0, \alpha, \mu) = N_b\left(N_0, p_0' = p_0^{\mu/\mu'}, \alpha' = \frac{\alpha^\mu}{\mu'}, \mu'\right),$$

$$N_b(N_0, p_0, \alpha, \mu) = \frac{N_0}{N_0'} N_b\left(N_0', p_0' = \left(\frac{N_0'}{N_0}\right)^\alpha p_0, \alpha, \mu\right),$$

only two of the four parameters p_0, α, μ, N_0 are independent: the discontinuous effects of varying μ and N_0 can be related to the effects of varying p_0 and α. Thus, throughout the paper we will keep fixed $\mu(\mu = 5/2)$ and $N_0(N_0 = 40)$, and vary mainly p_0 and α. For instance, figure 5 plots the discontinuous phase transition line $p_{0c}(\alpha,\mu,N_0)$ in the (p_0,α) plane.

We will show the following:

(i) For values of $p_0 < p_{0c}(\alpha,\mu,N_0)$, the process initiated by phase I alone (N_0 independent bankruptcies) propagates autonomously to a macroscopic number of firms. For p_0 just below $p_{0c}(\alpha,\mu,N_0)$, the crisis propagation starts at a fast pace but slows down around the hang-up point. After this point is overcome, if there is no further intervention, the crisis finally takes off to diverge increasingly fast to macroscopic scales.

(ii) For larger values $p_0 > p_{0c}(\alpha,\mu,N_0)$, phase I is not sufficient to trigger a macroscopic crisis.
   - One may still achieve macroscopic crisis propagation if phase II brings by forced contagion the total number of bankruptcies N_f above a critical value $N_f > N_S(N_0, p_0, \alpha, \mu)$. In this case, the crisis propagates autonomously to macroscopic N_bs according to rules **A** and **B**, even after the exogenous pressure stops.
   - On the contrary, for $N_f < N_S(N_0, p_0, \alpha, \mu)$, the process stops at finite N_b soon after the exogenous crisis factors cease to act.
   The discontinuity point N_S is plotted in figure 6 as a function of p_0 (for fixed α, N_0 and μ).

(iii) As opposed to the usual percolation transition p_c, the transition between regimes (i) and (ii) is discontinuous: at $p_0 = p_{0c}(\alpha,\mu,N_0)$ the number of bankruptcies necessary to induce a macroscopic crisis jumps from $N_f = N_0$ to about $N_f \sim 4N_0$ (e.g. equation (15)). Moreover, p_{0c} is, of course, much higher (e.g. equation (14)) than the value p_c in the absence of individual→collective interaction. Also, the final state of the process is completely different from the usual percolation outcome.

We will evaluate theoretically $p_{0c}(\alpha,\mu,N_0)$ and $N_S(N_0, p_0, \alpha, \mu)$ using the scaling properties of the percolation clusters around the percolation critical density ρ_c. The theoretical analysis and the Monte Carlo simulations agree.

2.4. The iterative autocatalytic percolation process

In order to understand theoretically and evaluate numerically the critical values $p_{0c}(\alpha, \mu, N_0)$ and $N_S(N_0, p_0, \alpha, \mu)$, we will use our knowledge of the percolation critical behavior. More precisely, we will use the fact that we know that the mean size of the clusters conquered by the crisis diverges when the density of potential failures approaches the critical density ρ_c:

$$N(\rho) = N_0 \left(1 - \frac{\rho}{\rho_c}\right)^{-\gamma}. \tag{3}$$

The coefficient N_0 is fixed by the requirement that, for $\rho = 0$, the clusters remain at the initial size $N(\rho = 0) = N_0$.

Equation (3) can be rewritten according to equation (1) also as

$$N = N_0 \left[1 - \left(\frac{p_c}{p}\right)^\mu\right]^{-\gamma}. \tag{3'}$$

Equations (3) and (3′) express the effects of the individual↔individual interactions in the model and emphasize the stochastic distributed character of the causality in the present model: the size of the bankruptcy clusters N depends on the individual fragilities (through their density ρ) and on their network of interactions (through ρ_c and γ).

Equation (3) is, of course, only an approximation for at least three reasons:

- As the clusters grow, they will start to merge and their number will decrease.
- For small clusters, far away from the scaling regime, the scaling is not guaranteed.
- We will see that there is significant non-Gaussian noise in the clusters, size distribution of the individual configurations realizing (3) and (3′).

This is why we will check this assumption directly by confronting the theory with the Monte Carlo simulations. In fact, we will explicitly confront below (figures 3 and 4) the scaling formula (3′) with Monte Carlo data and in this way fix the effective value of the critical exponent γ for the relevant range of parameters.

The fact that the only network information necessary to give a diagnostic and an intervention policy/strategy are the global parameters ρ_c and γ indicates that, while the present analysis refers for definiteness to a 2D square lattice, the method applies straightforwardly to any network for which ρ_c and γ can be obtained. The results are thus expected to hold for a wider range of network geometries.

Together with equation (1) (collective→individual) and equation (2) (individual→collective), equation (3) (individual↔individual) closes the autocatalytic loop:

$$p(N_0) \xrightarrow{\text{eq. (1)}} \rho(p(N_0)) \xrightarrow{\text{eq. (3)}} N_1 = N(\rho(p(N_0))) \xrightarrow{\text{eq. (2)}} p(N_1) \xrightarrow{\text{eq. (1)}} \rho(p(N_1)) \rightarrow \cdots$$

The closure of this chain clearly shows that the dynamics of the system evolve in autocatalytic cycles. Thus we can characterize the dynamics in terms of a series of cycles indexed by an integer k. Each cycle k consists of three steps:

- the decrease in global index p_k as a result of the previous cycle bankruptcies N_{k-1} (according to equation (2));
- the increase in the density of the number of potential bankruptcies ρ_k as a result of the new index p_k (according to equation (1));

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- the increase in the number of actual bankruptcies N_k as a result of the increase in the density ρ_k of potentially bankrupt agents (according to equation (3)).

Thus, according to equations (1)–(3), the theoretical analysis reduces to the study of the iterative process that represents the cyclic iteration of the three steps:

$$p_k = p(N_{k-1}) = p_0 N_{k-1}^{-\alpha}, \tag{4}$$

$$\rho_k = \rho(p_k) = p_k^{-\mu}, \tag{5}$$

$$N_k = N(\rho(p(N_{k-1}))) = N_0 \left(1 - \frac{\rho_k}{\rho_c}\right)^{-\gamma}. \tag{6}$$

Or, substituting ρ_k in N_k,

$$p_k = p_0 N_{k-1}^{-\alpha}, \tag{7}$$

$$N_k = N_0 \left[1 - \left(\frac{p_c}{p_k}\right)^{\mu}\right]^{-\gamma}, \tag{8}$$

with the initial value $N(0) = N_0$, i.e. $p_1 = p_0 N_0^{-\alpha}$.

Or, further substituting p_k in N_k,

$$N_k = N_0 \left[1 - \left(\frac{p_c}{p_0}\right)^{\mu} N_{k-1}^{\alpha\mu}\right]^{-\gamma}. \tag{9}$$

We will alternatively use forms (7), (8) or (9) to study the properties of the autocatalytic percolation process according to convenience.

3. Theoretical predictions

3.1. Graphical analysis

In order to visualize the iterative autocatalytic percolation process of equations (7) and (8), we represent in figures 1 and 2 the state of the system at iteration k as a point of coordinates (N_k, p_k) in the plane with coordinate axes N (horizontal) and p (vertical).

If curves (2) and (3′) intersect (figure 2) or are at a tangent, the values of N and p where this happens are fixed (stagnation) points of the process (7)–(8).

Consequently, the question of the divergence to macroscopic scales of the bankruptcy crisis process (7)–(8) reduces to the question of whether the system (2)–(3′) or, equivalently (substituting (2) into (3′)), whether the equation

$$N = N_0 \left\{1 - \left[\frac{p_c}{p_0}\right]^{\mu} N^{\alpha\mu}\right\}^{-\gamma} \tag{10}$$

has or does not have real solutions.

We call equation (10) the *fixed point Stauffer equation* in honor of one of the outstanding pioneers of percolation and of its application to social, biological and economic systems.

For the particular case $\gamma = 1/\alpha\mu$, the Stauffer equation (10) is analytically solvable and the two solutions $N_{\text{stop}}(N_0, p_0, \alpha, \mu)$ and $N_S(N_0, p_0, \alpha, \mu)$ are written explicitly in the next section (equations (12)–(13)) as well as condition (14) for these solutions to be real.

Before we compare the numerical theoretical predictions of the present section with the Monte Carlo simulations, we present the analytic solutions of the Stauffer equation for the analytically solvable case $\gamma = 1/\alpha\mu$.
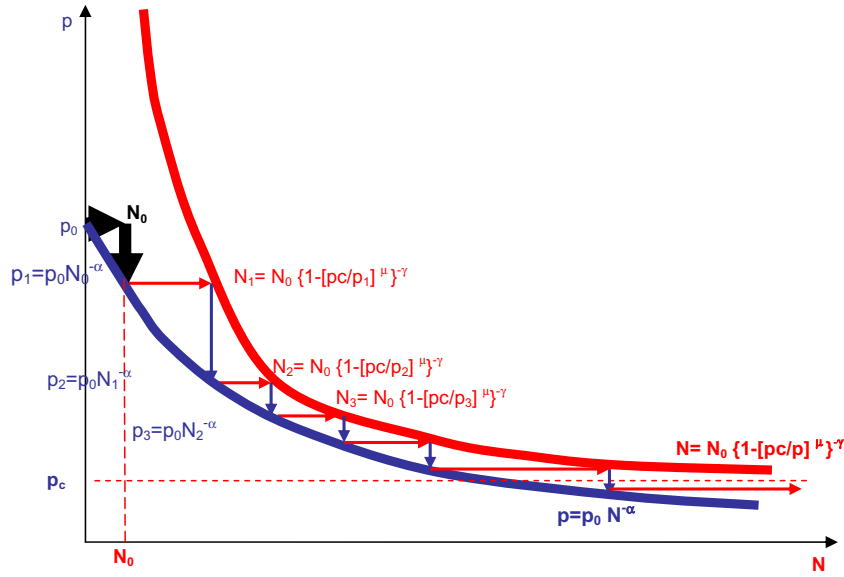
Figure 1. Evolution of the system under the influence of the iterative cycle equations (7) and (8) (a,b). The state of the system at iteration k is represented by the point of coordinates (N_k, p_k) in the plane with coordinate axes N (horizontal) and p (vertical). The (*blue*) vertical projections from the *red* line (equations (3) and (8)) to the blue line (equations (2) and (7)) represent the decrease in p due to the increase in N according to equation (7) (step a). The (*red*) horizontal projections from the blue line (equations (2) and (7)) to the red line (equations (3) and (8)) represent the increase in N due to the decrease in p according to equation (8) (step b). If, as in figure 1, the *red* curve $N = N_0[1 - (p_c/p)^\mu]^{-\gamma}$, (equation (3)) is always above and to the right of the *blue* curve $p = p_0 N^\alpha$ (equation (2)), the process leads to N increasing without any obstruction to macroscopic values (and p decreasing correspondingly). In other words, figure 1 illustrates the case when the process (7)–(8) (equivalently, the process equation (9)) has no fixed point and equation (10) (equivalently, system (2)–(3′)) has no real solution. See figure 8 for actual realizations of this scenario.

3.2. Analytical predictions

We obtained in the previous section the recursive formula (9) for the iterative growth of clusters of bankruptcies N_k and the Stauffer equation (10) for its fixed points.

For $\gamma = 1/\alpha\mu$, the Stauffer equation (10)

$$\left(\frac{N}{N_0}\right)^{-1/\gamma} = 1 - \left(\frac{p_c}{p_0}\right)^\mu N^{\alpha\mu},$$
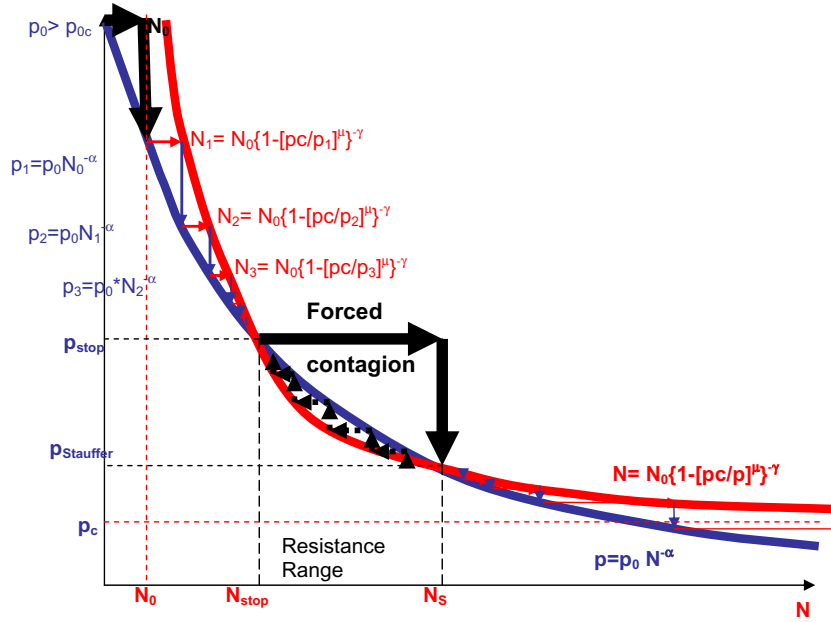
$$N^{-1/\gamma} N_0^{1/\gamma} = 1 - \left(\frac{p_c}{p_0}\right)^\mu N^{\alpha\mu}$$

reduces to an algebraic second-degree equation in $N^{1/\gamma}$,

$$N_0^{1/\gamma} = N^{1/\gamma} - \left(\frac{p_c}{p_0}\right)^\mu N^{2/\gamma} \tag{11}$$

and thus it can be solved analytically.

Figure 2. Evolution of the system with the same conventions as in figure 1. This time one illustrates the case $p_0 > p_{0c}$. The difference is that, for the case of figure 2, the process (7)–(8) (equivalently equation (9)) has two fixed points, i.e. the Stauffer equation (10) (equivalently, the system (2)–(3′)) has two solutions N_{stop} and N_S. One sees that, in this case, the bankruptcy propagation process that has only N_0 bankruptcies at phase I stops after a short while at N_{stop}. Even if one has a phase II with $N_f < N_{stop}$ bankruptcies exogenously induced by forced contagion, the result does not change: the process stops at $N_b = N_{stop}$. If N_f is in the interval (N_{stop}, N_S), then the resulting global index $p = p_0 N_f^\alpha$ (equation (2)) leads to a density $\rho = p_0^\mu N_f^{\alpha\mu}$ of potential bankruptcies, which in turn corresponds by equation (3′) to bankruptcy clusters of size $N = N_0[1 - (p_c/p)^\mu]^{-\gamma}$ lower than N_f, and thus the process stops immediately after the exogenous pressure ceases at N_f. The crisis will reach macroscopic scales only if there is a phase II that brings by forced contagion the number of bankruptcies to a value $N_f > N_S$. In this case, the process advances towards a macroscopic number of bankruptcies, even after the exogenous pressure ceases. See figure 9 for the actual realization of the scenario described by figure 2.

Having this analytic solvable case makes it easier to interpret the above generic analysis, as illustrated in figures 1 and 2.

- The lower solution of the fixed point equation (11) is:

$$N_{stop}(N_0, p_0, \alpha, \mu) = \left\{ \frac{1}{2} \left[\left(\frac{p_0}{p_c} \right)^\mu - \sqrt{\left(\frac{p_0}{p_c} \right)^{2\mu} - 4 \left(\frac{p_0}{p_c} \right)^\mu N_0^{1/\gamma}} \right] \right\}^\gamma. \tag{12}$$

If N_{stop} is real, the process starting at N_0 will stop spontaneously at N_{stop} if there is no initial exogenous forced percolation phase II.

- The upper fixed point N_S is

$$N_S(N_0, p_0, \alpha, \mu) = \left\{ \frac{1}{2} \left[\left(\frac{p_0}{p_c}\right)^{\mu} + \sqrt{\left(\frac{p_0}{p_c}\right)^{2\mu} - 4\left(\frac{p_0}{p_c}\right)^{\mu} N_0^{1/\gamma}} \right] \right\}^{\gamma}. \tag{13}$$

For the crisis to become macroscopic, one needs a forced contagion phase II until the number of bankruptcies $N_f > N_S(N_0, p_0, \alpha, \mu)$.

- The critical p_0 value,

$$p_{0c}(N_0, \alpha, \mu) = p_c 4^{1/\mu} N_0^{1/\gamma\,\mu} = p_c 4^{1/\mu} N_0^{\alpha}, \tag{14}$$

has the following interpretation. If the initial state of the system is $p_0 < p_{0c}(N_0,\alpha,\mu)$, then the expression in the square root of equation (12) (and (13)) is negative and there are no real fixed points. In this case, the process initiated by just phase I would propagate throughout the entire system self-sustained, even in the absence of the 'forced percolation' phase II.

Once p_0 reaches the critical value $p_0 = p_{0c}(N_0,\alpha,\mu)$, there is suddenly a real solution of (10) at

$$N_{\text{hang-up}}(N_0, p_0, \alpha, \mu) = 2^{-1/\alpha\mu} \left(\frac{p_0}{p_c}\right)^{1/\alpha} = 2^{\gamma} N_0. \tag{15}$$

So, at $p_0 = p_{0c}(N_0,\alpha,\mu)$, there is a serious jump in the number of bankruptcies necessary to destabilize the system from N_0 to $N_f \sim 4N_0$ (since $\gamma \sim 2$, cf Hoschen *et al* 1979, Margolina *et al* 1984).

However, if p_0 remains just below $p_{0c}(N_0,\alpha,\mu)$, the initial N_0 bankruptcies of phase I are enough to trigger a macroscopic crisis. One would still have a very long period of crisis 'incubation' around $N_b \sim N_{\text{hang-up}}$.

4. Monte Carlo simulations

4.1. Numerical experiments

The Monte Carlo simulation that realizes our autocatalytic percolation model is described by the following features.

Due to the two invariance properties, only two of the four parameters N_0, p_0, α and μ need to be varied. We generated random configurations by a Monte Carlo procedure. More exactly, we extracted independently for each site i of a 200×200 square lattice a random number p_i according to the probability distribution (1) with $\mu = 5/2$. For each configuration, we made runs starting with a few different random sets of initial $N_0 = 40$ bankrupt sites.

There are, in principle, a few possible simulation procedures realizing the model. In particular, one of the decisions to be made is at which stage, after updating the number of bankruptcies according to **A** and **B**, one updates the global index p according to equation (2).

There is no problem in updating p after each bankruptcy. However, this would be quite unrealistic since the global dynamics have shorter time scales than the individual firm.

Another possibility would be to apply **A** and **B** to all firms currently neighboring the bankruptcy clusters and then, according to the number of them that went bankrupt, to update p.

Yet another possibility is to keep p unchanged after the first wave of bankruptcies (composed of the first neighbors of the bankrupt clusters) and use the same p while one applies **A** and **B** to all new neighbors created (iteratively) by the new bankruptcies. Only when there are no such neighbors left would one take note of the current value of N_b and update the p index accordingly to $p = p_0 N_b^{-\alpha}$.

While we ran all of these alternatives, we prefer the last one (we call it the mesoscopic algorithm) from the point of view of relevance to the real world and for modeling convenience. The iterations generated in this way have a direct interpretation in terms of the cluster dynamics of the process: after each iteration k, the value of N_k corresponds to the size of the bankruptcy clusters at the value p_k that the global index has during that step.

This set of points (N_k, p_k) is the Monte Carlo correspondent of the points (N_k, p_k) obtained from the iteration of the theoretical formulae (7) and (8).

However, there are some differences between the Monte Carlo runs and the theoretical analysis based on (7) and (8).

For instance, in the recursive process (7) and (8), it is possible to trace the points residing between N_{stop} and N_S by just running the recursion backwards. In the Monte Carlo procedure, if one forces the system into a state $N_b = N_f \in [N_{\text{stop}}(N_0, p_0, \alpha, \mu), N_S(N_0, p_0, \alpha, \mu)]$ where the bankruptcy clusters size N_b exceeds in size the clusters corresponding (through equations (3′) and (8)) to the current index p, the process simply stops.

Thus, based on the theoretical analysis, the Monte Carlo runs are expected to display the following features.

One expects that there exists for each N_0, α and μ a critical value $p_{0c}(N_0, \alpha, \mu)$, such that for $p_0 < p_{0c}(N_0, \alpha, \mu)$ the process diverges (even in the absence of phase II) to values of N_b of size M of the system (figure 7).

We find p_{0c} by running, for various values of p_0, runs that have only phase I. We record as p_{0c} the largest value p_0 for which the process does not stop (figures 8 and 10).

For $p_0 \sim p_{0c}(N_0, \alpha, \mu)$, the crisis has a long period of indecision around $N_{\text{hang-up}}$. In this region, the number of new bankruptcies at each step is very small and the slightest noise or accident in the p_is or their positions can stop the process (figures 8 and 10).

If the process does not stop in the $N_{\text{hang-up}}$ region, it eventually diverges to macroscopic sizes in an exceedingly accelerated path.

For larger $p_0 > p_{0c}(N_0, \alpha, \mu)$ values (figure 9), one expects the following:

- there exists a value $N_b = N_{\text{stop}}(N_0, p_0, \alpha, \mu)$ at which the bankruptcy contagion process stops for any $N_f < N_{\text{stop}}(N_0, p_0, \alpha, \mu)$ (including $N_f = N_0$, i.e. in the total absence of phase II);

- there exists a value $N_b = N_S(N_0, p_0, \alpha, \mu)$ such that, for $N_f > N_S(N_0, p_0, \alpha, \mu)$, N_b continues to grow and reaches macroscopic values;

- if phase II ends between those values, $N_{\text{stop}}(N_0, p_0, \alpha, \mu) < N_f < N_S(N_0, p_0, \alpha, \mu)$, then the process ceases immediately after $N_b = N_f$. Thus in order to find $N_S(N_0, p_0, \alpha, \mu)$, one performs runs with different N_f and singles out as $N_S(N_0, p_0, \alpha, \mu)$ the lowest N_f for which the process still diverges.

These predictions are fulfilled very well by the Monte Carlo runs, but the amount of noise for each particular realization is quite large. To understand and correct it, we initiated a few further simulations and numerical experiments. These experiments clarify and strengthen our theoretical control on the system, as seen below. However, it will remain true that, for individual

realizations, policy makers should expect significant noise with the characteristics that we discuss in our analysis (see figure 10). Life realizes itself only once, and not exactly as ideally planned.

4.2. Noisy dependence of the bankruptcy clusters size N_b on p and the smooth 'lower envelope'

Let us discuss here the special source and character of noise in the percolation processes. It is known that the bankruptcy percolation clusters N_b do not grow smoothly with increase in the potential failures density ρ. This is trivially true in the neighborhood of the singularity around ρ_c, where (cf (3)) very small changes in the density ρ or in the economic index p may lead (cf (3′)) to macroscopic changes in the bankruptcy clusters N_b. However, even in the ρ ranges where the average cluster size given by equation (3) and (3′) changes smoothly with ρ or p, there is great discontinuity in the dependence of the particular cluster sizes on ρ or p.

To understand this, let us note that one cannot picture the percolation process as a gradual increase in the radius of a set of more or less regular clusters of bankrupt sites. Rather, the true mechanism is *clusters fusion*: the bankrupt clusters are surrounded by quite a number of clusters of potential failures. These clusters may have sizes comparable to the bankrupt one. Once a bankruptcy cluster touches even one site of a potential failure cluster, the entire potential failure cluster becomes bankrupt and joins the bankruptcy cluster. Thus, occasionally, upon a very small decrease of p (increase in ρ), N_b may increase by a quantity that in principle may be of the same order of magnitude as its own current size (or, to be more precise, $1/N_0$ of it).

Thus, the noise in the actual distribution of p_i values in the network is amplified to the noise in the cluster structures, and then amplified further in the way clusters join into larger clusters. In particular, of greatest dynamical impact are firms/sites with the lowest p_i. For instance, the clusters' neighboring sites with $p_i \in (p_{k+1}, p_k)$ are responsible for the cessation of the percolation process at the end of iteration k and for the extent of its continuation during the $k + 1$ iteration. In fact, equation (3′) captures in average exactly this increase of cluster size as a function of current p.

In order to identify and characterize these points, we adopted the following '*lower envelope*' procedure.

We ran Monte Carlo runs in the **A+B-modified** regime (i.e. the invasion percolation algorithm). In this way we obtained a rather noisy series of p_is (black points in figure 3). Out of the many values appearing in that series, we selected the 'lower envelope': that is the points that constituted real obstructions in the continuation of the process. More precisely, we singled out points whose p_i values were lower than any of the previous p_is that appeared in the series. The size N_b of the bankrupt clusters corresponding to each of these values was recorded. Subsequently, each 'lower envelope' point l was characterized by its coordinates (N_l, p_l) and plotted in the (N, p) plane.

This series of points is the Monte Carlo analogue of formula (3′) in as far as it connects the size of the clusters to the value of the index p, which leads by percolation to such clusters.

In fact, in figure 3 one sees a very good agreement between the two (the discrete yellow circles and the red continuous line, respectively).

Their relation is much like the relation between the series produced by the theory-based iterations (7), (8), and the runs based on the mesoscopic algorithm (**A** and **B** rules).

The difference is that, in the latter case, both the theoretical (red diamonds in figures 7–9) and the simulation series (green circles in figures 7–9) are sparse (only some values of $N = N_k$
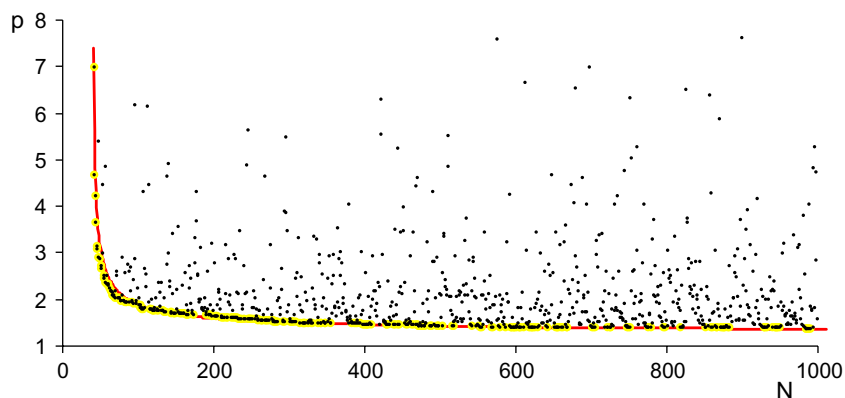
Figure 3. Black points represent, in order of their appearance (indexed by $N = N_b$), the values of p_i during a process generated by iterating steps (A) + (B) modified in a single Monte Carlo configuration. The initial number of exogenously enforced bankruptcies was $N_0 = 40$. One notices the noisy character of this p_i series. The yellow circles represent points of the 'lower envelope': the subset of p_is that are smaller than all the p_is preceding them in the series. In contrast to the initial p_i series, the lower envelope is quite regular (but sparse). One sees that the lower envelope fits well the scaling function equation (3') (and consequently also (8)), represented by the red continuous curve.
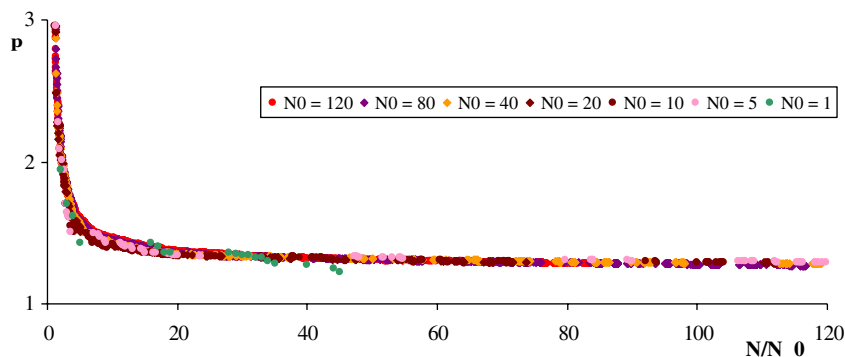
Figure 4. Comparison of the lower envelopes for different values of N_0 (but the same Monte Carlo configuration of p_is). One sees that the lower envelopes for different N_0s collapse on the same curve if one makes the rescaling $N \rightarrow N/N_0$). However, the points for small N_0, especially $N_0 = 1$ (green circles), are sparser and noisier. Thus the scaling equation 3' (and consequently also (8), (9)) is confirmed by the measurements on actual Monte Carlo generated configurations.

occur in the series). By contrast, in the present case the theoretical formula (3') is a continuous function (red line in figure 3) while the 'lower envelope' series is rather sparse (yellow circles in figure 3 and black points in figures 7–9). The density of points represented in the envelope increases with N_0, as shown by figure 4. The sparseness in the lower envelope implies large noise in the estimation of N_{stop} and N_S for individual configurations.

While in real-life situations these phenomena are unavoidable and there is no way to circumvent them, in numerical studies we can improve the precision of the confrontation of the Monte Carlo data with the theoretical predictions by the following procedures:

- We can increase N_0. Figure 4 demonstrates the efficiency of this procedure (in as far as the number of points on the lower envelope increases).

- An even more effective procedure is to overlay the lower envelopes obtained from various runs performed on different configurations. By plotting on the same graph the lower envelope points obtained from different runs (configurations), one can fill in the gaps and get a more densely populated curve. In the case where the same value of N is represented in different runs, one may even take their average and improve the precision and decrease the noise. In our case, just combining the lower envelopes from three different Monte Carlo configurations of p_is leads to satisfactory precision. The resulting 'superposed lower envelope' is then a more precise numerical counterpart of equation ($3'$) that connects theoretically the size N of the bankruptcy clusters to the global economic index p.

  According to the latter procedure, we used the superposed lower envelope to obtain higher precision numerical estimations of p_{0c}, N_{stop} and N_S by studying its intersection (or tangency) with the blue 'individual$\rightarrow$collective function' (2). The results obtained in this way are closer to the theoretical intersection between the continuous functions (2) and ($3'$) than the numerical realizations of individual runs, which sometimes have noise up to 30%. As seen in figures 5 and 6, and discussed in the next section, the results obtained by using the superposed lower envelope of just three independent configurations (green circles) are in excellent agreement with the theoretical predictions (red diamonds). Because the source of errors is the residual sparseness in the superposed lower envelope, there is no simple way to evaluate the errors: the plotted points are just the points on the envelope that are closest to the blue curve (2).

- Another, more drastic, way to avoid errors due to the 'holes' and noise in the Monte Carlo simulation results that connect the size of the bankrupt clusters N to the state of the economy p is to interpolate the (N, p) points of the superposed envelopes by a function of the form ($3'$) (we call it 'the envelope fitting function'). Such a fit constitutes a connection to the percolation scaling theoretical model and a direct validation of the scaling hypothesis ($3'$). In our case, superposing three configurations for each value of the parameters proved enough to give a very small standard deviation (Sigma ~ 0.1023) of the points of the envelope from a fit to ($3'$) with $\gamma \sim 2.1$.

Once the value of γ was established, we computed the intersection of this function ($3'$) with 'blue' individual$\rightarrow$collective interaction curves (2) corresponding to various p_0 and α. In this way, we obtained the orange square values of $p_{0c}(N_0, \alpha, \mu)$ and $N_b = N_S(N_0, p_0, \alpha, \mu)$ in figures 5 and 6. Thus, we could compare those results obtained with the envelope fitting function to the Monte Carlo simulations (in the form of the superposed envelope) and with the theoretical results provided by equations (7) and (8). As discussed in the next section, the graphs (figures 5 and 6) show very good agreement between the three methods.

More precisely, we have the following:

- The red diamonds are obtained from the condition that the process (7)–(8) (or, equivalently, (9)) hangs up at some stage ($N_{\text{hang-up}}$). This is also equivalent to the condition that equation (10) (equivalently the system (2), ($3'$)) has a single real solution.
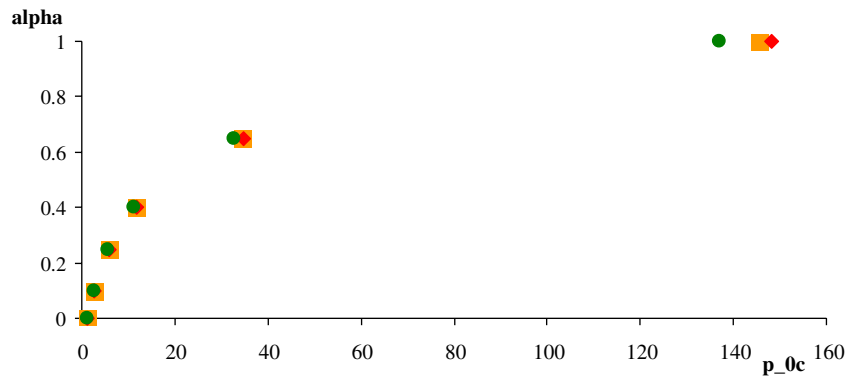
Figure 5. Discontinuous transition line between macroscopic crisis and macroscopic stability in the (α, p_0) plane. The points above and to the left of this line represent states of the system in which the macroscopic crisis is triggered by just phase I ($N_0 = 40$ exogenously induced bankruptcies). The points in the (α, p_0) plane below and to the right of this line represent states of the system that are stable under such exogenous strain. One compares the predictions obtained from the theoretical model (red diamonds) with the ones obtained from the Monte Carlo data (green circles) and from their fit (orange squares).
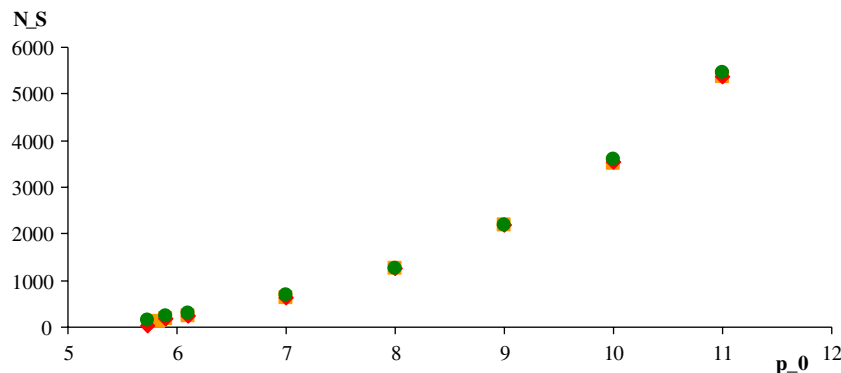
Figure 6. Dependence of the discontinuous transition point N_S on the initial state of the system p_0 for fixed $\alpha = 1/4$ and $N_0 = 40$, $\mu = 5/2$. One compares the predictions obtained from the theoretical model (red diamonds) with the ones obtained from the Monte Carlo data (green circles) and from their fit (orange squares).

- The green circles are obtained by requiring that the superposed lower envelope (of three independent configurations) meets the blue line (equation (2)) at only one point, whereas all other points of the envelope are above it (as in figure 8).
- The orange squares are obtained by requiring that the blue line (equation (2)) is at a tangent to the scaling function (equation (3′)) that fits the superposed lower envelope (as the red line in figure 3).

See figures 8 and 10 for an illustration of a typical situation generating the points in figure 5.
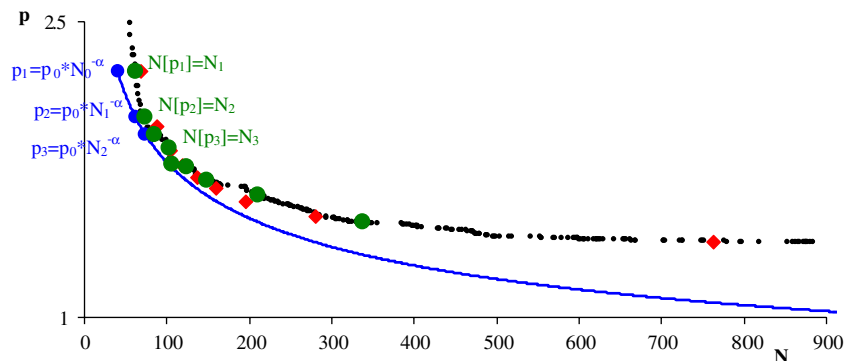
Figure 7. Various theoretical and Monte Carlo realizations of the scenario described in figure 1 for a particular choice of parameters ($\alpha = 1/4$ and $N_0 = 40$, $\mu = 5/2$ and $p_0 < p_{0c}$). To keep the figure simple, the (red) horizontal and (blue) vertical projections, as well as the scaling red line (3′) of figure 1, are not shown here. The blue continuous line represents the individual→collective interaction function (2). The red diamonds are the (N_k, p_k) points obtained from cycle k of the theoretical iterative process (7) and (8). The green circles represent the (N_k, p_k) points for each cycle k of the iterative Monte Carlo algorithm run iterating the **A** and **B** steps on a particular configuration with the same values of α, N_0, μ and p_0. One sees that the theoretically predicted red diamonds fall on the same line as the Monte Carlo obtained green circles. They also fall on the 'lower envelope' of the configuration (represented by black points).

More precisely, we have the following:

- The red diamonds are obtained by looking for the smallest starting value N_f for which the iterative process (9) (equivalently system (7)–(8)) still diverges.
- The green circles are obtained from the intersection (in figure 9) of the superposed lower envelope (such as the yellow points in figure 3) with the blue line (equation (2)).
- The orange squares are the intersection of the blue line (equation (2)) with the scaling function (equation (3′)) that fits the superposed lower envelope (as the red line in figure 3).

See figure 9 for an illustration of a typical situation generating the points in figure 6.

4.3. Comparing theoretical predictions with simulation results

In the present section, we compare the results of all of the methods that we used to study the autocatalytic percolation model. We think of the agent-based formulation of the model as the standard one, whereas all other formulations (recursive formulae, analytical equations, lower envelope, superposed lower envelope and envelope fitting function) are useful approximations to extract the relevant features.

Here, we compare the results from all the methods and find satisfactory agreement. For some the agreement is not surprising, but for others it contains nontrivial information about the reliability of the assumptions and the possibility of their future application to real-life problems.
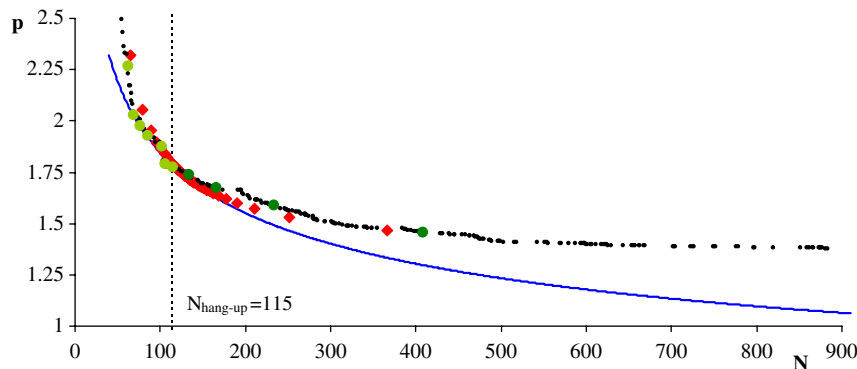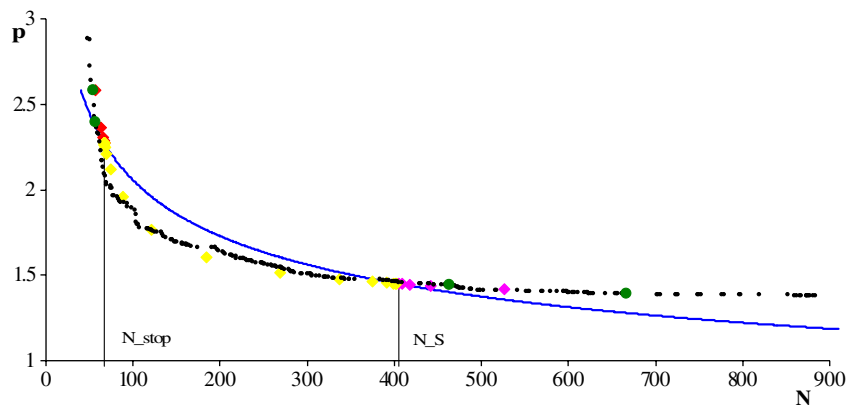
Figure 8. Various theoretical and Monte Carlo realizations of the tangent solution for equation (10). The choice of all other parameters, color conventions, the Monte Carlo configuration, N_0 seeds and the black points of the 'lower envelope' are identical to figure 7. The blue curve (2) differs only in the value of the factor p_0, which is taken such that $p_0 \sim p_{0c}(\alpha, \mu, N_0)$. More precisely, p_0 is chosen such that one single (black) point $N_{\text{hang-up}} = 115$ of the lower envelope falls on the blue curve (2). Thus, the Monte Carlo run (based on A and B iterations) is 'undecided' between stopping at $N_{\text{hang-up}} = 115$ (light green circles) or continuing beyond it (dark green) to ∞. This is further illustrated in figure 10. The theoretical iterations (7)–(8) represented by the red diamonds behave similarly: they (slow down or even) stop in the region where they (approach) touch the blue line (2).

Figures 7–9 are the realization of the processes described abstractly in figures 1 and 2. More precisely, they contain the results of iterative steps (N_k, p_k) in the case of percolation, in the case where there are obstructions to the free propagation of the crisis and in the limit case separating between them.

Figures 7–9 show, in the same framework, all the methods that we have been using so far in order to understand, interpret and control our autocatalytic percolation model. The green circles are the result of the Monte Carlo procedure (mesoscopic algorithm), the red diamonds are the result of the recursive iterative process (7) and (8), and the black points are the result of the lower envelope for a particular configuration. As illustrated in these figures, there is a very reasonable degree of agreement within the methods, even when only one configuration is used to represent the Monte Carlo method. However, as will be shown and explained in the next section, the outcomes from a single configuration exhibit occasional noisy departures from the theoretical predictions.

We can now discuss in more detail the results displayed in figures 5 and 6. These figures compare the values of the parameters separating regions of economic stability from regions of the parameters where the crisis propagates throughout the entire system.

Figure 5 refers to crises that were triggered only by a phase I: the exogenous bankruptcy of N_0 independent firms. In this case, in the region below and at the right of the points plotted in figure 5, the system is stable (the crisis does not percolate). This corresponds to the situations described in figures 2 and 9. For the parameter ranges above and at the left of the points plotted in figure 5, the system continues to collapse indefinitely after the first N_0 bankruptcies. This corresponds to the situations described in figures 1 and 7.

Figure 9. Various theoretical and Monte Carlo realizations of the scenario described in figure 2. The choice of all other parameters, color conventions, the configuration of p_is, N_0 seeds and the black points of the 'lower envelope' are identical to figures 7 and 8. The blue line (2) differs from figures 7 and 8 only in the value of the factor p_0, which is taken (as in figure 2) such that $p_0 > p_{0c}$. Consequently, system (2)–(3') has two real solutions N_{stop} and N_{S}, which correspond to the intersections of the blue (2) and red (3') curves of figure 2. N_{stop} and N_{S} are reliably reproduced in figure 9 by the (black) points of the lower envelope closest to the blue curve (2). The red diamonds are obtained from the theoretical iterative process (7)–(8) started at the initial N value $N = N_0$. As seen in the figure, the iterations converge to $N = N_{\text{stop}}$. The pink diamonds are obtained from the same process (7)–(8) but starting at N_{f} just above N_{S}. As one sees in the figure, this run diverges towards $N = \infty$. The yellow diamonds are obtained by starting the (7)–(8) iterations just below N_{S}. This theoretical run does not correspond to a real percolation simulation because the iterations result in N *decreasing* towards N_{stop}. Indeed, the green circles obtained by iterating the Monte Carlo process A and B cover only outside the stability range $(N_{\text{stop}}, N_{\text{S}})$. The run initiated by a phase I starting at N_0 covers the range (N_0, N_{stop}), while the run with a phase II with an N_{f} just above N_{S} diverges to $N = \infty$. One sees that the theoretically predicted diamonds fall on the same line with the Monte Carlo obtained green circles. The values for N_{stop}, N_{S} obtained by solving the system (2)–(3'), from the theoretical recursion (7)–(8), from the Monte Carlo runs **A** and **B** and from the intersection of the lower envelope series with curve (2) are all in good agreement. However, for precise comparison (figures 5 and 6) of the theory with the Monte Carlo numerical results, one should use the superposed envelopes of a few configurations rather than a single lower envelope series.

The points marked in figure 5 correspond to figure 8:

- The red diamonds in figure 5 are obtained by checking which is the highest p_0 for which the theoretical iterative process (7) and (8) (represented by the red diamonds in figure 8) still diverges.
- The green circles were obtained by finding the values of α and p_0 for which the individual→collective interaction function (2) touches the superposed lower envelope at just one point (as the black point at $N = 127$ in figure 8). We used a superposed lower
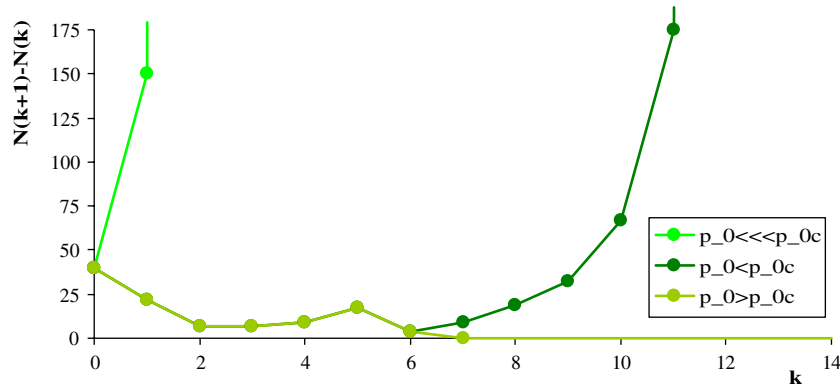
Figure 10. This figure exemplifies a situation where the slightest change in p_0 or in the value of p_i at $N_{\text{hang-up}} = 115$ has the effect of either stopping the process (light green circles) or ensuring its continuation to macroscopic scales (dark green circles). In fact, the values of p_0 for the two runs are so close that, for the first six iterations after the initiation of the crisis by $N_0 = 40$ exogenously induced bankruptcies, the light and dark green circles coincide. The behavior starts differing at the 7th cycle, when $N_{\text{hang-up}} = 115$ is either contaminated or not, depending on an infinitesimal change in its p_i or p_0 (see figure 8). Following this, in the run where $p_i > p_0 115^{-1/4}$, the number of bankruptcies diverges to ∞, while in the run where $p_i < p_0 115^{-1/4}$, the bankruptcies cascade stops.

envelope obtained by overlying the lower envelopes of three Monte Carlo generated configurations.

- The orange squares were obtained by requiring that the blue line equation (2) is at a tangent to the envelope fitting function of the form (3′) that best fits the superposed lower envelope. More precisely, we used the fitting function (3′) with an exponent γ that minimizes the square standard deviation of the points of the superposed lower envelope (as the red line fits the yellow points in figure 3).

Figure 6 describes the stability of the system in the presence of an exogenous initial stress that includes a phase II that leaves the system with N_0 bankruptcy clusters totalling N_{f} bankruptcies. For values of N_{f} lower than the points plotted in figure 6, the system is still stable (there is no crisis percolation). For values of N_{f} larger than the ones plotted in figure 6, the crisis spreads throughout the entire system. More explicitly, we have the following:

- The red diamonds are obtained by determining, for each p_0, which is the lowest starting point $N_{\text{f}} = N_{\text{S}}$ for which the theoretical iterative process (7)–(8) still diverges (similar to the divergent chain of vertical and horizontal projections on the right side of figure 2).

- The green circles are the intersections of the blue individual→collective interaction curve (2) with the superposed lower envelope of three particular configurations. This is similar to the intersection of the blue curve with the black points series in figure 9 (except that in figure 9 the black points are obtained from only one envelope rather than the superposing three). Actually, since the envelope is a discrete series, its closest point to curve (2) is taken as N_{S}.

- The orange squares are obtained as the intersection of the blue individual→collective interaction curve (2) with the curve of form (3′) that interpolates best (minimizes the standard deviation to) the superposed lower envelope. This is similar to the way in which we obtained the N_S point in figure 2.

The agreement between the p_{0c} and N_S values obtained in figures 5 and 6 by various theoretical and Monte Carlo procedures (as well as the agreement on the results displayed by figures 7–9) suggests that our modeling framework can be used to predict the crisis percolation scenarios based on the knowledge of the general properties of the economic agents' connectivity (3), the individual→collective interactions (2) and the collective→individual effects (1).

5. Stability of the system and policy implications

The processes that we study in the present paper are of quite a special type. On the one hand, they do not have the linear deterministic dynamics normally used in economics and financial narratives. On the other hand, nor are they at the statistical mechanics extreme where very robust averages over large ensembles are considered.

The triggering of a crisis and the propagation of the crisis contagion cannot be described by a clear chain of individual causes and effects. Yet the number of individual events and agents that can dramatically affect the outcome of a crisis can be too small to admit an average treatment. In order to use the tools offered by the present approach, the policy makers will have to make a paradigm shift and accept the concept of distributed causality. More precisely, it makes no sense to ask which of the N_0 initial failures is responsible for the crisis reaching systemic size. It also does not make sense to ask which of the N_f forced contagion failures that take the process beyond the Stauffer stability range is the key to systemic collapse. It is clear that not all of the agents are equally responsible: agents on the 'lower envelope' are clearly more likely to become, under the slightest intervention, obstructions to bankruptcy wave propagation.

In fact, one sees from figure 8, and from figure 10, that a rather limited number of points play crucial roles in the unfolding of the crisis. Thus, on top of the rather precise statistical characteristics of the systems, there are a few events and agents (e.g. around $N_{hang-up}$) that can change quite dramatically a particular crisis history. For instance, in figures 8 and 10, one sees that infinitesimal changes in one p_i (e.g. p_{115}) can make the difference between the process stopping or diverging.

From these examples, one sees that the detailed study of the stability region is necessary to assess system stability against crisis percolation, and to defend a stable system against exogenous attacks or catastrophes.

Indeed, a system with an economic index $p_0 < p_{0c}(\alpha, \mu, N_0)$ has no Stauffer stability range and no chance to survive: even one accidental bankruptcy would throw it into a self-sustaining fast propagating autocatalytic percolation process. In order to save the economy, policy makers have to intervene immediately if $p_0 < p_{0c}(\alpha, \mu, N_0)$, even if the naive static percolation stability condition $p_0 > p_c$ is fulfilled (here, p_c corresponds to a potential bankrupt firm's density equal to the critical percolation density). This is, of course, because the naive static percolation threshold ρ_c does not take into account the individual→collective equation (2) influence of the bankruptcies on the global index p and the resulting autocatalytic loop.

Once the policy makers ensure that at all times the initial global index $p_0 > p_{0c}(\alpha, \mu, N_0)$, one is assured that there exists a Stauffer region of stability. This is a very important tool to

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defend system stability under exogenous attacks/catastrophes/crises. Indeed, on the one hand one has to identify whether the value N_0 for which $p_0 = p_{0c}(\alpha,\mu,N_0)$ is not so dangerously small as to make it realistically realizable. On the other hand, even if one has incurred a number N_0 of exogenously induced bankruptcies for which the safety condition $p_0 > p_{0c}(\alpha,\mu,N_0)$ is fulfilled, one has to take care that the external conditions do not induce forced contagion of bankruptcies to their partners (neighboring sites). In particular, one has to be particularly cautious when the total number of bankruptcies N_f approaches $N_S(N_0,p_0,\alpha,\mu)$. As long as one is in the Stauffer stability region, one can still stop the process by very limited intervention that neutralizes the exogenously forced contagion of the neighbors of recently bankrupt sites.

This opportunity should be grabbed in time, because later many firms may become endogenously potential failures and their salvaging may become less justified, more costly and ultimately logistically impossible (if there are too many of them).

There are other positive conclusions from the existence of the Stauffer stability region. As we have seen, even if a bankruptcy contagion wave continues to propagate autocatalytically self-sustained for quite a while (sometimes up to $4N_0$), this does not necessarily mean that the macroscopic crisis is on its way: it could be that the wave will slow down and stop by itself as one approaches $N = N_{stop}(N_0,p_0,\alpha,\mu)$. However, the mere slowing down of the contagion rate is not a guarantee that the percolation will stop: it may just indicate that one is in the slow region around $N_{hang-up}$ where the red (3′) and the blue (2) lines are close to one another or, in algebraic terms, that N is close to the real part of the possibly imaginary solutions of the Stauffer equation.

6. Conclusions

In the present paper, we have studied the implications of the action of the collective on its own individual components. This influence, sometimes termed top-down, has long been recognized in the social sciences, but its formal study has been elusive in the agent-based models for quite a while.

As opposed to physical systems, where the interactions are usually individual↔individual in social and economic systems, the individual→collective and collective→individual effects are significant (Solomon *et al* 2000, Hohnisch *et al* 2008, Cantono and Silverberg 2009).

In the present paper, we have developed a method to express quantitatively and study analytically, numerically and by simulation the effects of the autocatalytic loop generated by the influence of the collective on the individual states, and of the individual states on the collective.

This autocatalytic loop is responsible for the emergence and resilience of complex collective macroscopic phenomena out of simple elementary interactions. Moreover, the autocatalytic loop selects a limited number of parameters as relevant to the macroscopic behavior of the system. We have chosen as an illustration of this mechanism the social/market percolation framework, as presented in recent works (Antonelli 1996, Goldenberg *et al* 2000, Bornholdt 2001, Aleksiejuk and Holyst 2001, Weisbuch *et al* 2001, Erez *et al* 2005, Goldenberg *et al* 2005, Yaari *et al* 2006, Erez *et al* 2007, Frenken *et al* 2008, Hohnisch *et al* 2008, Sieczka and Holyst 2009, Lorenz *et al* 2009). We consider here the example of the economic crisis and bankruptcy contagion. We give policy suggestions about how to forecast, prevent, control and eventually stop the propagation of a distress to systemic scales. In order to address these issues quantitatively, our model includes the autocatalytic feedback loop

between individual→collective, collective→individual and individual↔individual effects. The integration of these features into a recursive autocatalytic percolation cycle leads to a complete reconsideration of real-life policy implications. In the present paper, we have provided the theoretical framework necessary for achieving it using agent-based modeling and analytical results. We have tested analytical and numerical predictions with Monte Carlo simulations and found good agreement among all the methods. However, the processes that we have studied are of a quite special type and require careful interpretation at the level of specific realizations. On the one hand, they do not have the linear deterministic dynamics usually used in economics and financial narratives. The triggering of a crisis and the propagation of the crisis contagion cannot be described by a linear chain of individual causes and effects. On the other hand, nor are they at the statistical mechanics extreme where only robust averages over large ensembles are considered: the number of individual events and agents that can dramatically affect the outcome of a crisis can be too small to admit an average treatment. Indeed, we have shown how a limited number of elements of a system play a crucial role in the unfolding of the crisis.

We have also shown that, as opposed to the usual percolation transition that is continuous, in our model the system's resistance undergoes discontinuous transitions between a few possible regimes. Thus, depending on the initial state of the system, the interactions and heterogeneity of its elements, and the extent of the disturbance, one experiences discontinuous jumps between the following regimes:

- If the disturbance acts only upon a few isolated elements, then the system still has a chance to survive, provided that the propagation of the disturbance is not too widespread. It might be reasonable at this stage to hamper the propagation either by rehabilitating the isolated falls or by reinforcing the nearest neighbors.

- Unfortunately, additional pressures may drive the system to a forced propagation of the disease anyway. Yet, this same disease may still be confined to a limbo, either by the resilience of the system itself or by pushing the general state of the system upwards.

- However, under certain circumstances, the valleys of tears can percolate and overflow to flood the entire system. Without external intervention, the propagation of the crisis is inevitable.

Emergency measures, such as those needed in this situation, fall into two categories: very expensive and general, or efficient and specific, the latter being feasible provided that enough information is available.

In future, we will validate the generic use of this kind of theoretical model to the real propagation of contagion in actual financial and economics systems.

Acknowledgments

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